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Lecture 20.

- 'Usual Functions : $f \; \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}$
- Sequences : $a_1, a_2, \ldots$

for every $n \in \mathbb{N}$, real number are $\mathbb{R}$

This is a function $\mathbb{N} \to \mathbb{R}$, $n \longmapsto a_n$

(Sequences are functions too).

Sequence of functions

$$f_1, f_2, \ldots$$

Example $f_n(x) = x^n$ on $[0, \infty)$. $n \in \mathbb{N}$

$f_1(x) = x$

$f_2(x) = x^2$

$f_3(x) = x^3$

$\vdots$

Example 2 $f_n(x) = nx$

$f_1(x) = x$

$f_2(x) = 2x$

$\vdots$

Definition (Pointwise Convergence).

We say f_n converges pointwise to f if for every $x \in \Omega$ we have that $f_n(x) \to f(x)$

$\varepsilon - \delta$: Fix $x \in \Omega$. Let $a_n = f_n(x)$ and let $a = f(x)$

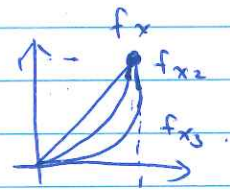
$a_n \to a$.

For every $\varepsilon > 0$, $\exists \, \mathbb{N}$ s.t. $|a - a_n| < \varepsilon$, $\forall \, n > N$

Example : $f_n(x) = x^n$ $x \in [0, \infty)$

converges for $x \in [0, 1]$. (pointwise).

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases}$$



Note $f_n(x)$ is continous on $[0,1]$, but the limit $f(x)$ is discontinuous.

This motivates us to give a different notion of convergence

$f_n(x)$ – continuous
$[0,1]$ – not continuous f.

Uniformly convergence

Definition: we say f_n convergence to f uniformly on Ω if for every $\varepsilon > 0$, $\exists N$ such that $|f(x) - f_n(x)| < \varepsilon$ for $\forall n > N$ and all $x \in \Omega$

Claim: $f_n(x) = x^n$ does not uniformly converge to

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

Suppose, for contradiction that it did uniformly converge. This means for every $\varepsilon > 0$, $\exists N$ s.t. $|f(x) - f_n(x)| < \varepsilon$ $\forall n > N$, $\forall x \in [0,1]$.

Take $\varepsilon = \frac{1}{4}$, so $\exists N$ such that $|0 - f_n(x)| < \frac{1}{4}$ for $x \in [0,1)$.
But $f_n(x)$ are continuous function so if $|f_n(x)| < \frac{1}{4}$ for all $x \in [0,1)$ then we have to have $|f_n(1)| \leq \frac{1}{4}$
This is a contradiction
So f_n does not unif. converge to f

But $f_n(x) \rightrightarrows f(x)$ for $x \in [0,p]$ for every $p < 1$

$\rightrightarrows$: uniformly convergent

Example $f_n \rightrightarrows f$ on $[0, \frac{1}{2}]$
Given $\varepsilon > 0$ we have to find N s.t.
$\underbrace{|f_n(x)|}_{x^n} < \varepsilon$ for $\forall n > N$, $\forall x \in [0, 1/2]$.

Note: max of x^n on $[0, \frac{1}{2}]$ is achieved at $1/2$

Choose N such that $(\frac{1}{2})^N < \varepsilon$
Then this N will work for all $x \in [0, \frac{1}{2}]$
because $|f_n(x)| = x^n \leq (\frac{1}{2})^n \leq (\frac{1}{2})^N < \varepsilon$ $\square$

Given ε, when is $(\frac{1}{2})^N < \varepsilon$?
$\Leftrightarrow \log(\frac{1}{2})^N < \log \varepsilon$ $\quad N < \frac{\log \varepsilon}{\log \frac{1}{2}}$
$N \log(\frac{1}{2}) < \log \varepsilon$

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~~Lecture 21~~

Uniform convergence $f_n \rightrightarrows f$

Example: $f_n(x) = x^n$

$f(x) = 0$.

Then f_n converges to f pointwise on $[0,1]$, but not uniformly. But $f_n \rightrightarrows f$ on $[0,p]$ where $p < 1$.

Lecture 22

Example $f_n(x) = (x(1-x))^n$ on $\mathbb{R}$.

$x(1-x) = 1$.

$\Leftrightarrow x - x^2 - 1 = 0$

$\Leftrightarrow x^2 - x + 1 = 0$

$|x(1-x)| = 1$

$\Leftrightarrow x(1-x) = \pm 1$

$$\Leftrightarrow \left.\begin{matrix} x - x^2 - 1 = 0 \\ \text{or} \quad x - x^2 + 1 = 0 \end{matrix}\right] \Leftrightarrow \begin{matrix} x^2 - x + 1 = 0 \\ x^2 - x - 1 = 0 \end{matrix}$$

$x = \frac{1 \pm \sqrt{5}}{2}$

↳ Golden ratio

→ Find when $f_n(x)$ is uniformly convergent.

/

Theorem: If $f_n : [a,b] \to \mathbb{R}$.
are continuous function and
$f_n \rightrightarrows f$ then f is also continuous

$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$
$\leq \underbrace{|f(x) - f_n(x)|}_{\mathrm{I}} + \underbrace{|f_n(x) - f_n(x_0)|}_{\mathrm{II}} + \underbrace{|f_n(x_0) - f(x_0)|}_{\mathrm{III}}$

Uniform convergence $f_n \rightrightarrows f$ means:
Given $\varepsilon > 0$, we can find N s.t.
$|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ for $n > N$ and $x \in [a,b]$

Note: $f_n : [a,b] \to \mathbb{R}$ continuous $\to$ unif. cont.
This means, given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if
$|x - x_0| < \delta$, then $|f_n(x) - f_n(x_0)| < \varepsilon/3$ for $\forall x$
and $x_0 \in [a,b]$. This gives us II. But III is also handled
by the above bound by taking $x = x_0$
So we can make I, II & III smaller than $\varepsilon/3$.

x_n converges. $\Leftrightarrow$ $\forall \varepsilon$, $\exists N$ such that if $k, l > N$ then
$|x_k - x_l| < \varepsilon$

Theorem ($f_n : \Omega \to \mathbb{R}$.)
f_n converges ~~to~~ uniformly to f $\Leftrightarrow$ $\forall \varepsilon > 0$,
$\exists N$ s.t. $|f_k(x) - f_l(x)| < \varepsilon$ $\forall k, l > N$,
$\forall x \in \Omega$

($\Rightarrow$) Suppose $f_n \rightrightarrows f$
Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t
$|f_k(x) - f(x)| < \varepsilon/2$. $\forall k > N$, $x \in \Omega$
So then, $|f_k(x) - f_l(x)| \leq |f_k(x) - f(x)| + |f(x) - f_l(x)|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\forall k, l > N$
$\forall x \in \Omega$

($\Leftarrow$) for all $x \in \Omega$
$\{f_n(x)\}$ is Cauchy.
So $f_n(x)$ converges
Let $f(x) = \lim_{n \to \infty} f_n(x)$
Then $f(x)$ is a function $\Omega \to \mathbb{R}$

Claim: $f_n \rightrightarrows f$
Given $\varepsilon > 0$, $\exists N$ s.t. $k, l > N$ then
$|f_k(x) - f_l(x)| < \varepsilon/2$ for all x.
Let $k \to \infty$ and fix l. $|f(x) - f_l(x)| \leq \frac{\varepsilon}{2} < \varepsilon$ $\square$

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Lecture 23

Summary Sequence of functions $\{f_n\}$.

$f_n : \Omega \to \mathbb{R}$

Limits of sequence of functions:

1. pointwise limit
2. Uniform limit

Example: $f_n(x) = x^n$, $f(x) = \begin{cases} 0 & \text{on } [0,1) \\ 1 & x = 1. \end{cases}$

$f_n \to f$ pointwise on $[0,1]$

↑ continuous . ↑ discontinuous.

Theorem: Uniform limit of continuous function is continuous; if $f_n \rightrightarrows f$ and f_n continuous, then so is f.

* Cauchy criterion for uniform convergence.

Sequence $\{a_n\}$, then you can build a series out of this by considering $\sum_{n=1}^{\infty} a_n$

series:

* $a_n \to 0 \not\Rightarrow \sum a_n$ converges (eg. $a_n = \frac{1}{n}$)

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let $S_k = \sum_{n=1}^{k} a_k$. Then we say that $\sum a_n$ converges if the sequence $S_1, S_2 \dots$ has a limit.

Series of functions

Let $\{f_n\}$ be a sequence of function

$f_n : \Omega \to \mathbb{R}$

We want to consider $\sum_{n=1}^{\infty} f_n(x)$

For each k, let $S_k(x) = f_1(x) + \dots + f_k(x)$

We say: ① $\sum_{n=1}^{\infty} f_n$ converges pointwise if $\{S_n\}$ converge pointwise.

② $\sum_{n=1}^{\infty} f_n$ converges uniformly on Ω if $\{S_n\}$ converges uniformly.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

Example : Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

Weirstrasse M-test.

Suppose $\exists\ M_n$ non-negative real number such that $\sum_{n=1}^{\infty} M_n < \infty$
And suppose $f_n : \Omega \to \mathbb{R}$ satisfies $|f_n(x)| \leq M_n$ for all $x \in \Omega$
Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Proof Let $S_k(x) = f_1(x) + \ldots + f_k(x)$
Because $\sum_{n=1}^{\infty} M_n < \infty$ for every $\varepsilon > 0$, $\exists\ N$ such that if $n > m \geq N$ then $\sum_{\ell=m+1}^{n} M_\ell < \varepsilon$.
(This is expressing the fact that partial sums for $\sum M_n$ is Cauchy)

Now we have. $|S_n(x) - S_n(x)| = \left|\sum_{\ell=m+1}^{n} f_\ell(x)\right|$

$\triangle$-ineq. : $\leq \sum_{\ell=m+1}^{n} |f_\ell(x)| \leq \sum_{\ell=m+1}^{n} M_\ell < \varepsilon$

So by Cauchy criteria, $S_n(x)$ converges uniformly $\square$

Example : Does $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converge?

Note: $\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}$

Now, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by Weistram test,

the above series converge

* Why does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

- Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = 1.$

↑ doesn't help.

* Is $\sum_{n=4}^{\infty} \frac{1}{n^2} < \sum_{n=4}^{\infty} \frac{1}{2^n}$?

* $\sum_{n=2}^{\infty} \frac{1}{n^2} < \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots$$

= Telescoping sum.